Math 254-2 Exam 2b Solutions

1. Carefully state the definition of "dependent", in the context of this course. Give 2 examples from $\mathbb{R}^2$.

   A set of vectors is dependent if there is a nondegenerate linear combination yielding the zero vector. Many examples are possible: $\{(1,2),(2,4)\}$ satisfies $2(1,2)-(2,4)=(0,0)$; $\{(1,2),(2,3),(3,4)\}$ has three elements while $\dim \mathbb{R}^2 = 2$, hence must be dependent.

The remaining problems all concern the following system:

$$\begin{aligned} 6x_2 - 2x_4 &= 4 \\ x_1 + 3x_2 - 2x_3 + x_4 &= 3 \\ 3x_1 + 3x_2 - 6x_3 + 5x_4 &= 5 \\ -2x_1 + 4x_3 + 10x_4 &= 12 \\ x_1 + 3x_2 - 2x_3 + 8x_4 &= 10 \end{aligned}$$

2. Write the above system as a matrix equation.

   This system, like all linear systems, can be written as $Ax=b$.

$$A = \begin{bmatrix} 0 & 6 & 0 & -2 \\ 1 & 3 & -2 & 1 \\ 3 & 3 & -6 & 5 \\ -2 & 0 & 4 & 10 \\ 1 & 3 & -2 & 8 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 12 \\ 10 \end{bmatrix}.$$

3. Write the above system as an augmented matrix. Put this in echelon form, justifying each step using elementary row operations. Using the echelon form, find all solutions to the system.

$$\begin{bmatrix} 0 & 6 & 0 & -2 & 4 \\ 1 & 3 & -2 & 1 & 3 \\ 3 & 3 & -6 & 5 & 5 \\ -2 & 0 & 4 & 10 & 12 \\ 1 & 3 & -2 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 1 & 3 \\ 0 & 6 & 0 & -2 & 4 \\ 0 & -6 & 0 & 2 & -4 \\ 0 & 6 & 0 & 12 & 18 \\ 0 & 0 & 0 & 7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 1 & 3 \\ 0 & 6 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14 & 14 \\ 0 & 0 & 0 & 7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 1 & 3 \\ 0 & 6 & 0 & -2 & 4 \\ 0 & 0 & 0 & 14 & 14 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   $\{-3R_2 + R_3 \to R_3, 2R_2 + R_4 \to R_4, -R_2 + R_5 \to R_5, R_1 \leftrightarrow R_2\}$,
   $\{R_2 + R_3 \to R_3, -R_2 + R_4 \to R_4\}, \{R_4 - 2R_5 \to R_5, R_3 \leftrightarrow R_4\}$

   $14x_4 = 14$, hence $x_4 = 1$. x_3 is free. $6x_2 - 2x_4 = 4$, hence $x_2 = 1$. $x_1 + 3x_2 - 2x_3 + x_4 = 3$, hence $x_1 = 2x_3 - 1$. The general solution is $(2x_3 - 1, 1, x_3, 1)$.

4. Write the above system as an augmented matrix. Put this in row canonical form, justifying each step using elementary row operations. Using the row canonical form, find all solutions to the system.

   We pick this up where the previous problem left off.

$$\begin{bmatrix} 1 & 3 & -2 & 1 & 3 \\ 0 & 6 & 0 & -2 & 4 \\ 0 & 0 & 0 & 14 & 14 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 1 & 3 \\ 0 & 6 & 0 & -2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & 0 & 2 \\ 0 & 6 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   $\{(1/14)R_3 \to R_3\}, \{(2R_3 + R_2 \to R_2, -R_3 + R_1 \to R_1\}, \{(1/6)R_2 \to R_2, -3R_2 + R_1 \to R_1\}$

   It is now easy to read off $x_1 = -1 + 2x_3, x_2 = 1, x_4 = 1$, yielding the same general solution.

5. Write the homogeneous system associated to the above system. Solve this homogeneous system. Then use the particular solution $(1,1,1,1)$ to give the general solution to the original system.

   The homogeneous system is $Ax = 0 = (0,0,0,0,0)^T$. Its augmented matrix is identical to the one above, except that the rightmost column is all zeroes.

$$\begin{bmatrix} 0 & 6 & 0 & -2 & 0 \\ 1 & 3 & -2 & 1 & 0 \\ 3 & 3 & -6 & 5 & 0 \\ -2 & 0 & 4 & 10 & 0 \\ 1 & 3 & -2 & 8 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ (with the same operations as above)}$$

   The general solution to the homogeneous system is $x_4 = x_2 = 0, x_1 = 2x_3$, i.e. $(2a, 0, a, 0)$. We add this to the specific solution $(1,1,1,1)$ to get the general solution $(1+2a, 1, 1+a, 1)$.